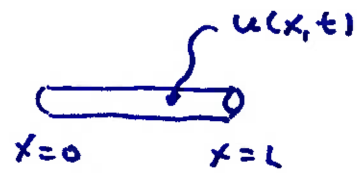
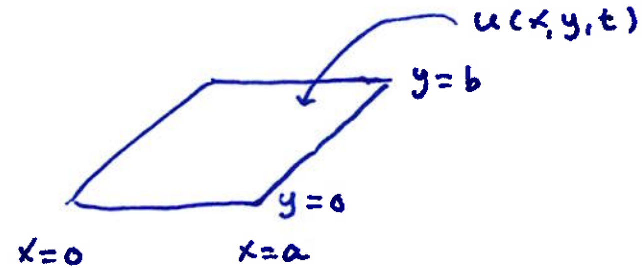
Higher-Dimension PDEs

1-D Heat eq: $u_t = k u_{xx}$ $\quad 0 < x < L$

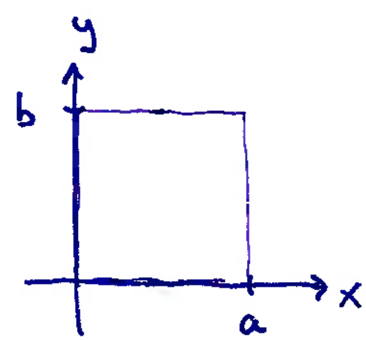


2-D Heat eq: $u_t = k(u_{xx} + u_{yy})$ $\quad 0 < x < a \quad 0 < y < b$



Set up: $u_t = k(u_{xx} + u_{yy}) \quad 0 < x < a \quad 0 < y < b$

BCs (four edges)



all four edges at $u=0$ (homogeneous)

$u(x,0,t) = 0$ (bottom)

$u(a,y,t) = 0$ (right)

$u(x,b,t) = 0$ (top)

$u(0,y,t) = 0$ (left)

IC: $u(x,y,0) = f(x,y)$

separation of variables : $u(x,y,t) = X(x)Y(y)T(t)$

$u_t = k(u_{xx} + u_{yy})$ becomes

$$XYT' = k(X''YT + XY''T)$$

$$\vdots$$

$$\underbrace{\frac{X''}{X}}_{\text{depends on } x \text{ along}} = \underbrace{-\frac{Y''}{Y} + \frac{T'}{kT}}_{\text{depends on } y \text{ and } t} = \text{constant} = -\lambda$$

ODEs: $\boxed{X'' + \lambda X = 0}$

$$-\frac{Y''}{Y} + \frac{T'}{kT} = -\lambda$$

rewrite: $$\underbrace{\frac{Y''}{Y}}_{\text{depends on } y} = \underbrace{\frac{T'}{kT} + \lambda}_{\text{depends on } t} = \text{constant} = -\mu$$ (another separation constant) more ODEs: $Y'' + \mu Y = 0$ (looks just like X eq.)

$\vdots$

$T' + k(\mu + \lambda)T = 0$ $X'' + \lambda X = 0$

BCs: $u(x,0,t) = 0 \rightarrow X(x)Y(0)T(t) = 0 \rightarrow Y(0) = 0$

$u(a,y,t) = 0 \rightarrow \cdots \rightarrow X(a) = 0$

$u(x,b,t) = 0 \rightarrow \cdots \rightarrow Y(b) = 0$

$u(0,y,t) = 0 \rightarrow \cdots \rightarrow X(0) = 0$

$X'' + \lambda X = 0$ $X(0) = X(a) = 0$ $\lambda_n = \dfrac{n^2\pi^2}{a^2}$ $X_n = \sin\left(\dfrac{n\pi}{a}x\right)$

$n = 1, 2, 3, \ldots$

$Y'' + \mu Y = 0$ $Y(0) = Y(b) = 0$ $\mu_m = \dfrac{m^2\pi^2}{b^2}$ $Y_m = \sin\left(\dfrac{m\pi}{b}y\right)$

$m = 1, 2, 3, \ldots$

$$T' + k(\mu+\lambda)T = 0$$

$$T' + k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)T = 0$$

$$\boxed{T_{mn} = e^{-k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)t}}$$

for each pair of (n,m) $\quad u_{mn} = T_{mn} X_n Y_m$

general solution: sum over m, n

$$\boxed{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, e^{-k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2}\right)t} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)}$$

IC: $u(x,y,0) = f(x,y)$ initial heat distribution

at $t=0$

$$f(x,y) = \sum_{m=1}^{\infty}\underbrace{\left[\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{a}\right)\right]}_{\text{"constant" if } x \text{ is fixed, call it "C"}}\sin\left(\frac{m\pi y}{b}\right)$$ (double Fourier series)

looks like: $\underset{(x \text{ fixed})}{f(x,y)} = \sum_{m=1}^{\infty} C \sin\left(\frac{m\pi y}{b}\right)$ "regular" sine series

$$C = \frac{2}{b}\int_0^b f(x,y)\sin\left(\frac{m\pi y}{b}\right)dy = \sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{a}\right)$$

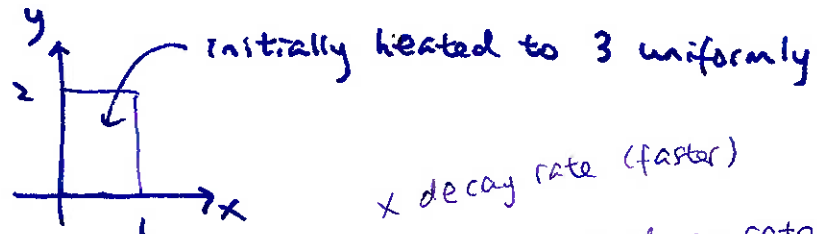
sine series again

$$A_{mn} = \frac{2}{a}\int_0^a \left[\frac{2}{b}\int_0^b f(x,y)\sin\left(\frac{m\pi y}{b}\right)dy\right]\sin\left(\frac{n\pi x}{a}\right)dx$$

$$\boxed{A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi x}{a}\right)dy\,dx}$$

<u>example</u> $a=1$, $b=2$, $f(x,y)=3$

$k=1$

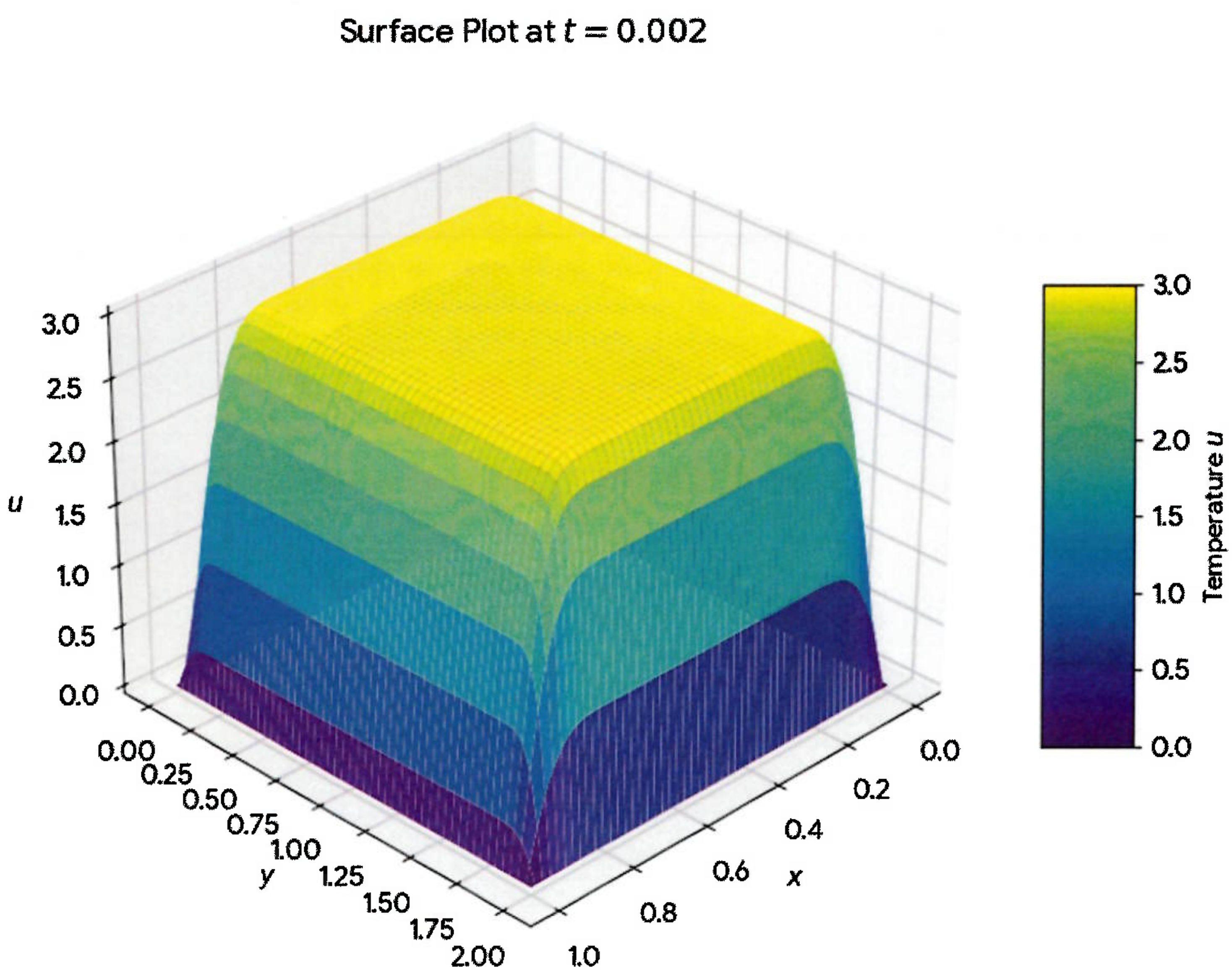


⋮

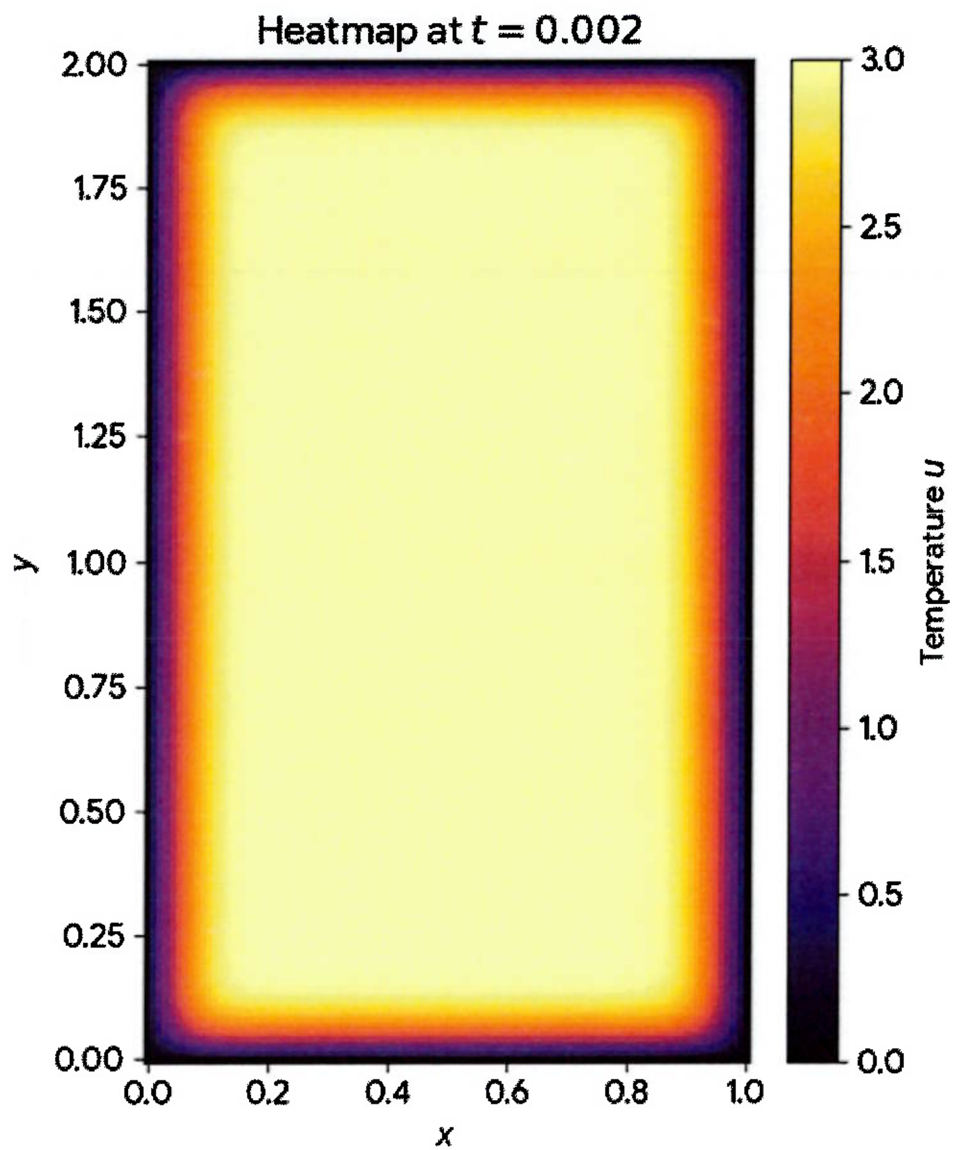
x decay rate (faster)

y decay rate

$$\boxed{u(x,y,t) = \sum_{m\ \mathrm{odd}}^{\infty}\ \sum_{n\ \mathrm{odd}}^{\infty} \frac{48}{mn\pi^2}\sin(n\pi x)\sin\left(\frac{m\pi}{2}y\right)e^{-\left(n^2\pi^2 + \frac{m^2\pi^2}{4}\right)t}}$$

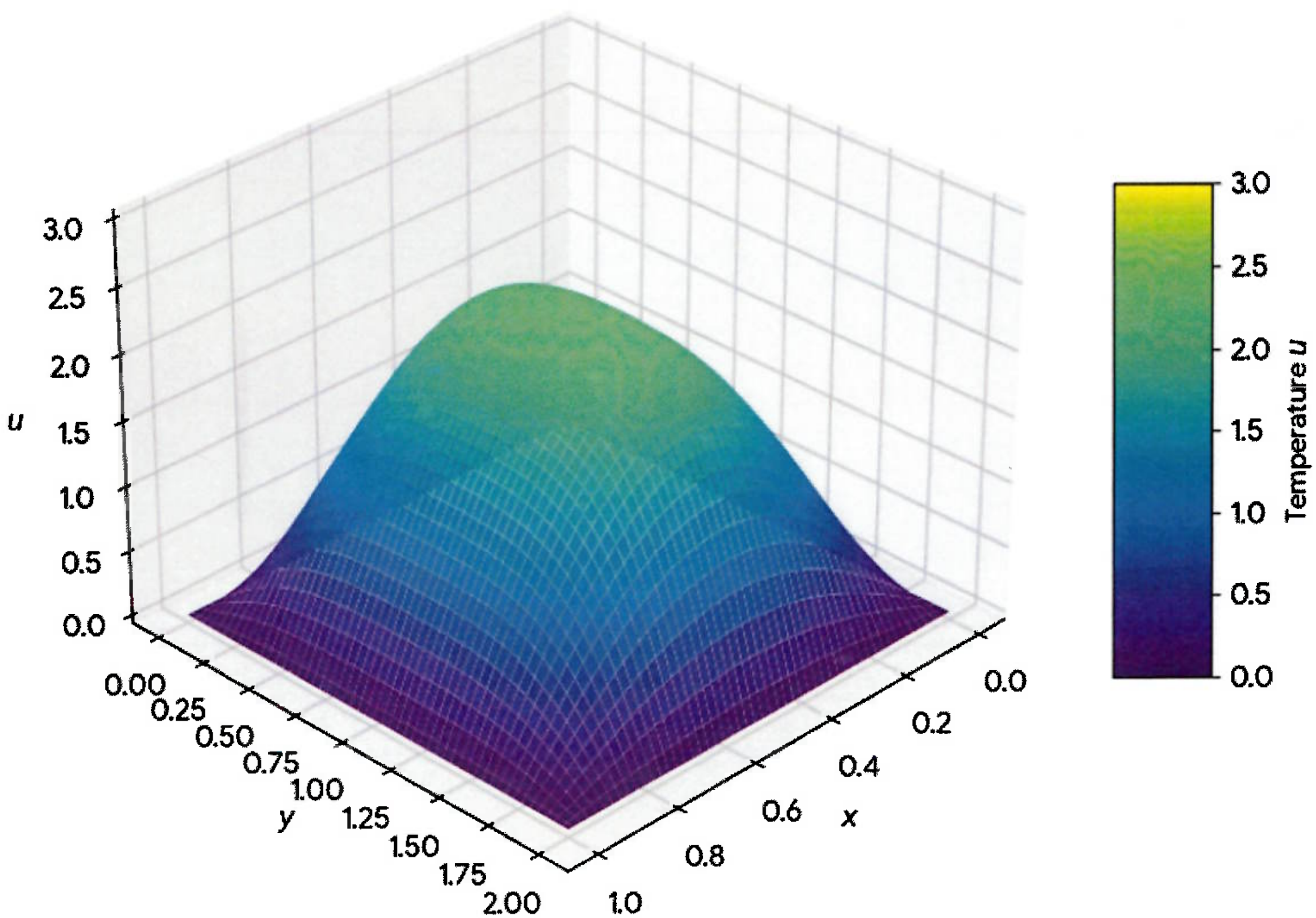


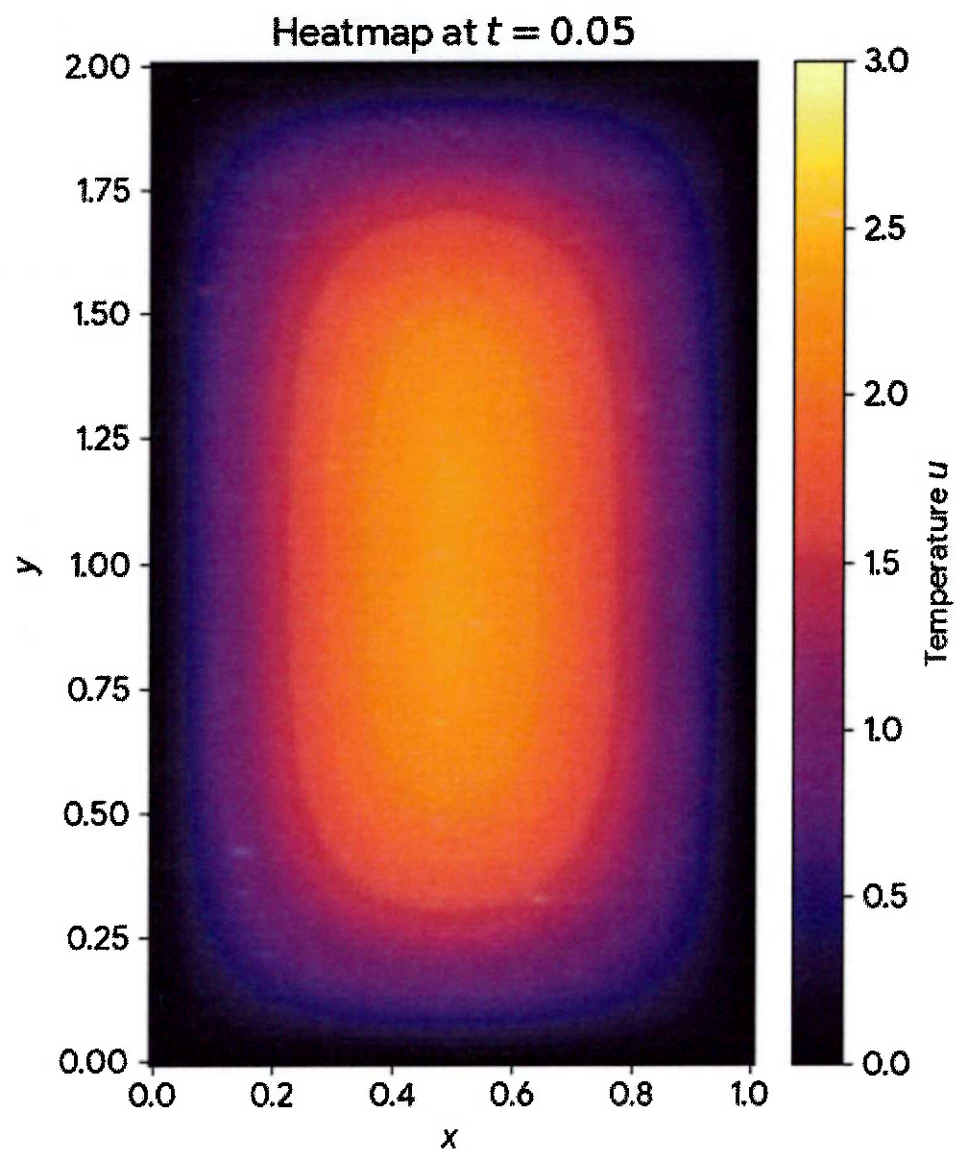
Surface Plot at $t = 0.002$
u
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
Temperature u



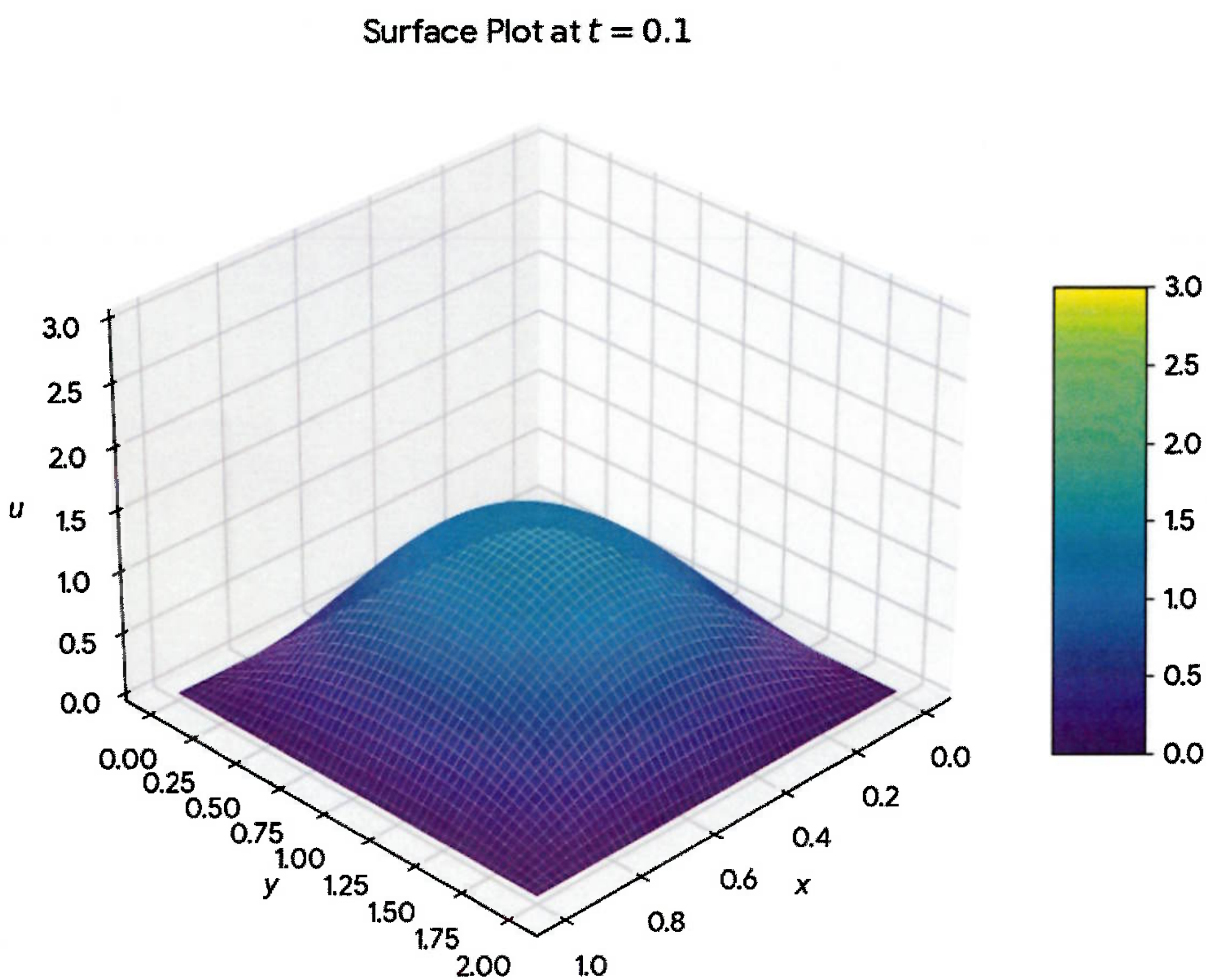
Heatmap at $t = 0.002$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u

Surface Plot at $t = 0.05$

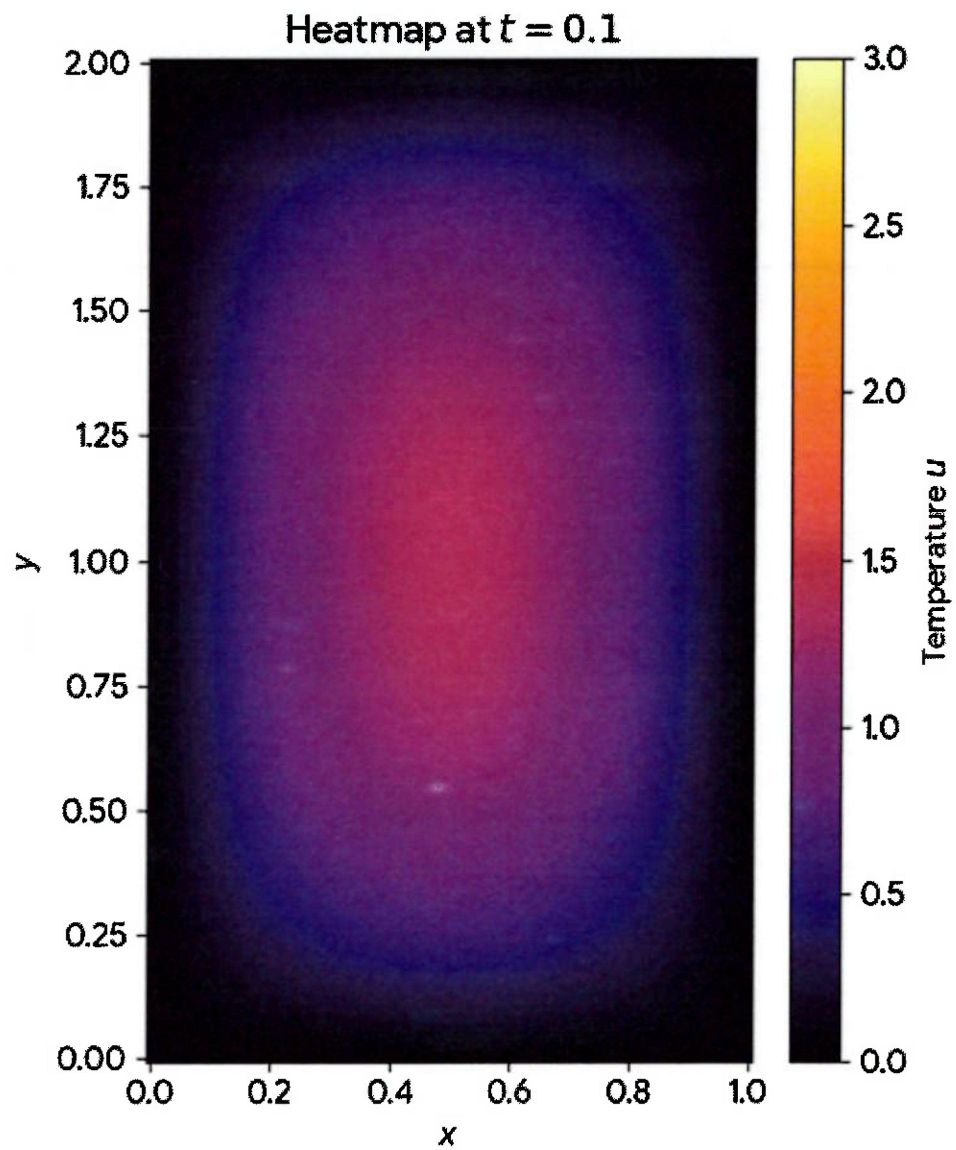




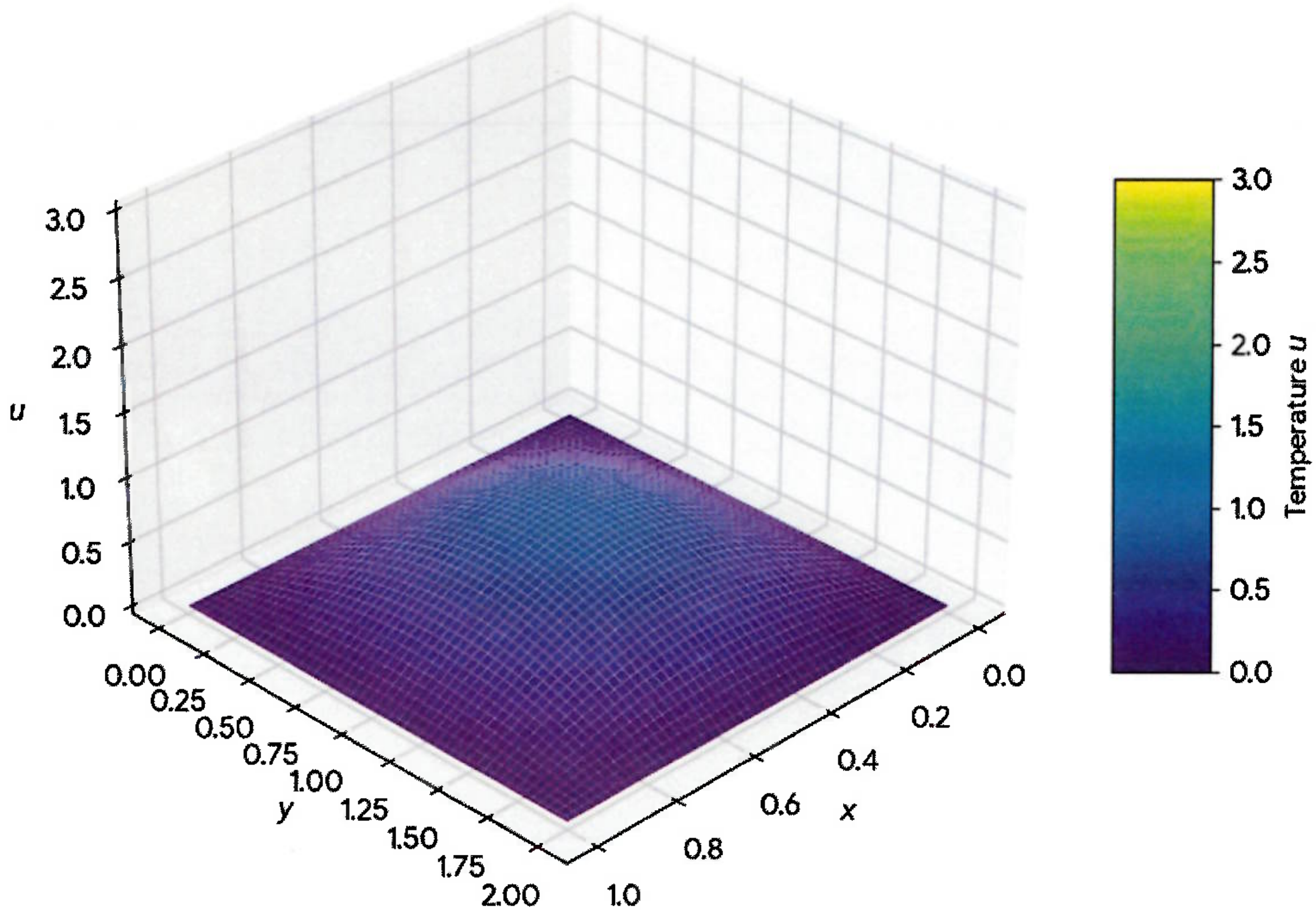
Heatmap at $t = 0.05$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u

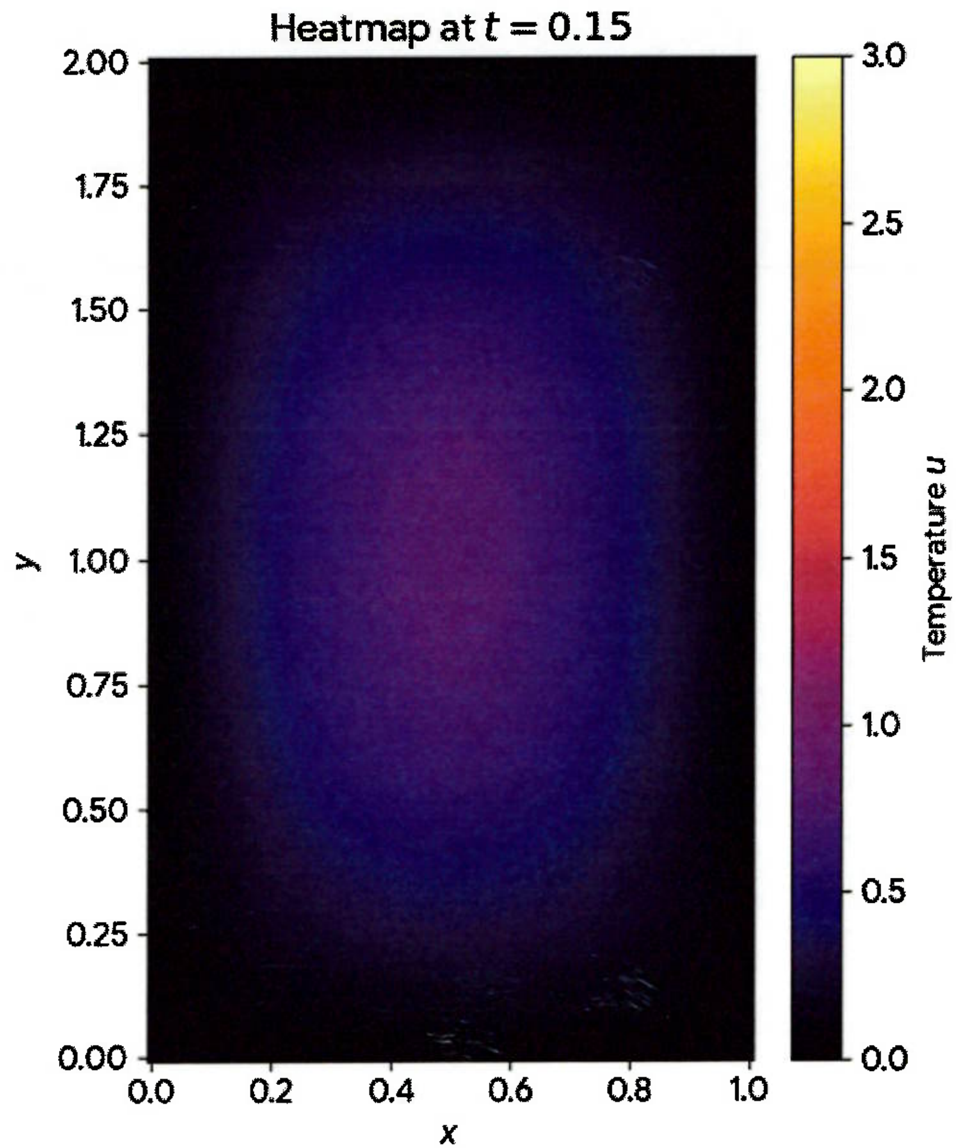


Surface Plot at $t = 0.1$
u
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
0.0
0.2
0.4
0.6
0.8
1.0
Temperature u


Heatmap at $t = 0.1$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u

Surface Plot at $t = 0.15$




Heatmap at $t = 0.15$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.0
0.2
0.4
0.6
0.8
1.0
x
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Temperature u